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Adverse Selection, Moral Hazard and Endogenous Matching in a Dynamic Assignment Model*

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Abstract

This paper formulates and analyzes a dynamic infinite horizon assignment model in which a firm repeatedly matches workers with tasks and output depends on the characteristics of the tasks, of the workers, and of the effort exerted by the workers. The characteristics of the workers and the effort they exert are unobservable, so there is both adverse selection and moral hazard (on one side). The paper studies the two-stage game in which the firm commits to matching workers to tasks and to payment on the basis of observed output and workers then exert effort produce output and receive payment in each period, and defines a notion of equilibrium for this game. At equilibrium, adverse selection is eliminated and moral hazard is mitigated. Firm profit in equilibrium is compared against natural benchmarks.

Key Words: dynamic assignment model, endogenous matching, adverse selection, moral hazard

1 Introduction

The seminal work of Shapley and Shubik (1971) on the assignment model has given rise to a vast literature on models of matching with transfers.[1] These models have found an enormous range of applications: to marriage markets (Becker, 1974), to labor markets (Shimer and Smith, 2000), to international trade (Grossman, Helpman and Kircher, 2014), to perfect competition (Gretsky, Ostroy and Zame, 1999), to hold-ups (Cole, Mailath and Postlewaite, 2001; Makowski, 2004). Most of this literature focuses on (the implications of) the "proper" matching of the two sides of the market – buyers to sellers, workers to firms, men to women, or, as is in our setting, workers to tasks – and on the division of output/value. Most of this literature treats a static, rather than dynamic, environment. Most of this literature assumes that characteristics are observable. Perhaps more remarkably, the entirety of this literature assumes that output simply falls from the sky – that producing output does not require costly effort – and entirely ignores the fact that it is not enough to match the correct worker to the correct task; it is also necessary to provide incentives for the worker to exert effort.

This paper addresses a dynamic version of the assignment model in which a single firm must repeatedly match tasks to workers, who must exert costly effort in order to produce output. Worker characteristics and effort are unobservable, so that there is both adverse selection and moral hazard (on one side). We formulate the situation as a two-stage game: in the first stage, the firm commits to a (specific) matching rule and a payment schedule; in each

[1] Of course, the seminal work of Gale and Shapley (1962) has given rise to a different but equally vast literature on models of matching *without* transfers.

period of the second stage the workers are matched to tasks, choose effort and produce, and payments are made. We identify a notion of equilibrium and show that (with natural assumptions), equilibrium outcomes eliminate adverse selection and mitigate moral hazard. The key to our framework is that the repeated matching of workers to tasks is determined *endogenously* on the basis of (observable) output.

Formally, we consider an infinite horizon discrete time environment with a single long-lived firm and a continuum of long-lived workers W. In each period, a continuum of tasks T arrives to the firm, which matches tasks to workers. If worker w is matched to task t and exerts effort e it produces output $Y(e, w, t)$, incurs a cost $C(e, w)$ and receives a payment P and so receives the net utility $U = P - C$; the firm earns the net profit $Y - P$. (The formal model imposes assumptions that render the analysis tractable; see Section 2.) The quality of tasks is commonly observed but characteristics of workers (productivity and cost) and of course effort, are not observed. Thus there is both adverse selection and moral hazard (on one side). We treat the simplest and most obvious endogenous matching rule: in each period, workers are assigned to tasks according to the ranking of the output they produced in the previous period. We define a notion of equilibrium that corresponds to a single committed choice of payment scheme by the firm and steady state behavior by workers in the ensuing (infinite horizon) subgame. For each choice of payment scheme, the worker's subgame has a unique equilibrium in which matching is perfectly assortative (better workers are matched to better tasks) so the firm's problem is to choose the payment scheme that maximizes profit. We show that the firm's profit is strictly greater than in either of two benchmarks: random matching of workers to tasks, with workers choosing effort optimally, or full information assortative matching of workers to tasks, with workers choosing effort optimally. Thus, the endogenous matching rule – that workers who produce more output are matched with better tasks – leads both to "proper" matching of workers to tasks and to stronger incentives for workers to exert effort. The firm benefits because it is able to provide incentives to workers *both* by paying for output *and* by conditioning future assignments of tasks on current output, so that workers who produce more in the current period are matched to better tasks in the next period.

In the general model, it does not seem possible to solve the model in closed form – even to solve for equilibrium in the workers' subgame – or to quantify the comparisons of firm profits in equilibrium and in the benchmarks. We therefore present examples in which solving in closed form and quantifying comparisons are possible; for these examples we can also determine the firm optimal (linear) payment schedule and determine how close firm profit in equilibrium comes to the profit the firm could achieve if it had full information and could tailor payment schedules to individual characteristics.

Several features of our model deserve special attention. The first of these features is that the matching between workers and tasks is endogenous, and this endogeneity is a crucial driver of our conclusions. By contrast, in more familiar repeated game environments, either the matching is fixed – the same players interact in each period – or the matching is random. In these environments, the actions of players today affect the way in which others will play against them in the future, but not the future matching; in our setting, the actions of workers today affect how they will be matched in the future.

The second of these features is that the dependence of output on effort, and hence the presence of moral hazard, has all the familiar implications – but it also has an important and unfamiliar implication: effort matters for output and hence for matching – but effort is endogenous. As we show by example, in a general setting the firm might prefer *not* to match workers to tasks assortatively; indeed, the firm might prefer random matching to assortative matching.

We are not aware of previous work on assignment models in which output depends on effort (in addition to the characteristics of workers and tasks) and so there seems no work that is very close to ours in economic terms. The work that seems closest to ours in mathematical terms is Hopkins (2012), which treats a static model in which output depends on the observable characteristics of firms and the unobservable characteristics of workers, so there is one-sided adverse selection. Hopkins' model generates a signaling game, the analysis of which leads, through the work of Mailath (1987), to a differential equation that resembles ours. Because we treat fully dynamic infinite horizon model, our analysis is much more subtle and complicated.

Following this Introduction, Section 2 describes the environment in which we work. Section 3 describes the matching rule. Section 4 describes our notion of steady state assortative equilibrium in the subgame played by the workers and Section 5 demonstrates existence and uniqueness of steady state assortative equilibrium in this subgame. Section 6 describes the notion of equilibrium in the full two-stage game and shows that equilibrium exists. Section 7 presents natural benchmarks and qualitative comparisons with these benchmarks. Section 8 computes closed form solutions and explicit comparisons for a class of examples. Section 9 offers a few concluding remarks. All proofs are collected in the Appendix.

2 Environment

We first introduce the basic framework of Tasks, Workers, Output, Cost and Payments and the various assumptions. As noted in the Introduction, we make assumptions that make the analysis tractable; we also offer some discussion about more general assumptions.

There is a fixed space of *tasks* $T = [B, 1]$, where $B \geq 0$. Task t is characterized by its *quality* $q(t)$; we assume that $q : T \to [0, \infty)$ is smooth (by "smooth" we always mean twice continuously differentiable) and (differentiably) strictly increasing and that $q(t) > 0$ if $t > 0$.[2] For convenience, we assume that the "population" of tasks is uniformly distributed and that the total mass of tasks is $1 - B$; this entails that the mass of tasks with quality less that that of task t is $(t - B)$. (The assumption that tasks are uniformly distributed according to their ranking by quality entails little loss of generality; the distribution of tasks by quality is implicitly defined by the function q.)

The space of *workers* is $W = [B, 1]$. Worker w is characterized by its *productivity* $p(w)$ and its *worker-specific cost factor* $k(w)$. We assume that $p : W \to [0, \infty)$ is smooth and (differentiably) strictly increasing and that $p(w) > 0$ if $w > 0$.[3] We assume that $k : W \to [0, \infty)$ is smooth and weakly

[2] In the examples, it is convenient to take $B = 0$ and assume $q(0) = 0$ so that the worst task is worthless.

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decreasing. (Thus better workers are more productive at lower cost.) For simplicity, we assume that the population of workers is uniformly distributed and that the total mass of workers is $1-B$ (so that there are the same number of workers as tasks); this entails that the mass of workers with productivity less than that of worker w is $(w-B)$. (Given the assumption that workers who are more productive also have lower cost factors, the assumption that workers are uniformly distributed according to their ranking again entails little loss of generality: the distribution of workers by productivity is implicitly defined by the functions p, k.)

If worker w is matched to task t and exerts effort $e \in E = [0, \infty)$ then it produces *output* $Y(e, w, t) = e\, p(w) q(t)$. (Given that output is multiplicatively separable, the assumption of linearity in effort is computationally convenient but innocuous. Workers choose effort e; if output were non-linear but strictly increasing and weakly concave in effort, so that $Y(e, w, t) = h(e)p(w)q(t)$, we could simply view workers as choosing *virtual effort* $h(e)$ rather than actual effort e. Linearity has the additional implication that, independent of the productivity of the worker and the quality of the task, aribtrarily large output can always be produced by exerting sufficient effort, but this fact plays no essential role.)

If worker $w \in W$ is matched to task $t \in T$, exerts effort $e \in [0, \infty)$ and produces output $Y = e\, p(w)q(t)$ it incurs the *cost* $C(e, w) = k(w)c(e)$ where $c : [0, \infty) \to [0, \infty)$ is a common *effort cost factor*.[4] We assume that c is

[4]We could allow for cost that depends on effort, worker type and task type, provided that we made additional assumptions on the common cost factor $c(e)$. However doing so would make the analysis more complicated without adding much.

smooth, strictly increasing and strictly convex; in order to guarantee that optimal choice of effort always exists (in particular, when workers behave myopically) we also assume that

$$\lim_{e\to 0}\frac{c(e)}{e}=0 \quad\text{and}\quad \lim_{e\to\infty}\frac{c(e)}{e}=\infty$$

In view of the assumed convexity of c, this is equivalent to assuming that $c'(0) = 0$ and $\sup c'(e) = \infty$.[5] Finally, we assume that marginal cost is *log-concave* with respect to effort; in view of our assumptions this means $c''(e)/c'(e)$ is weakly decreasing.

If the worker produces output y then it receives a payment $P(y)$ where $P : [0,\infty) \to [0,\infty)$ is the *payment schedule*. By assumption, payment depends only on output. This seems natural: output is observable but effort and worker type are not. We require that $P(y) \le y$ so that payment is a *share of output*. The realization of the payment is the *wage*, which will be determined endogenously in equilibrium. For simplicity we focus here on linear payment schedules $P(y) = \lambda y$, with $\lambda \in (0,1)$.[6]

As usual, we assume utility is quasi-linear, so if in a given period, worker receives payment P and incurs cost C its net period utility is $P - C$.

Workers discount future utility at the constant rate $\delta \in (0,1)$.[7] It is

[5] If the domain of possible effort choices were bounded, the latter form of the assumptions would be more convenient.

[6] Our analysis could be extended to payment schedules that are concave in output, and even to payment schedules that depend (multiplicatively separably) on output and on task type but again, the analysis would become significantly more complicated.

[7] The assumption that workers share a common discount factor is familiar but is unnecessary. As we discuss at the end of Section 5, we could allow for worker-specific discount factors.

convenient to normalize, multiplying discounted lifetime utility by $1-\delta$ and so expressing lifetime utility in terms of discounted average per-period rewards. Hence, if in each period n, the worker is matched to task t_n, exerts effort e_n and produces output y_n, its lifetime utility will be

$$(1-\delta)\sum_{n=0}^{\infty}\delta^n[P(t_n, y_n) - C(e_n, w)]$$

2.1 Implications of the Assumptions

It may be helpful to note the most important implications of these assumptions. For output Y the most important implications are that for $e, w, t > 0$ we have:

- $Y > 0$
- Y is continuously differentiable and strictly increasing in each variable
- Y is strictly supermodular in each pair of variables
- Y is weakly concave in effort

For cost C the most important implications are that

- C is continuously differentiable, strictly increasing and strictly convex in effort and weakly decreasing in worker type
- C is weakly submodular

For the interaction between output and cost the most important implication is that

- optimal effort exists and is strictly positive

Finally, there is an important implication for imputed effort. Write $\Phi(y, w, t) = y/[p(w)q(t)]$ for the effort required for worker w matched with task t to produce output y. (Note that Φ is well-defined and continuously differentiable when $w, t \neq 0$.) Direct calculation shows that

- the ratio $\frac{\partial\Phi/\partial y}{\partial\Phi/\partial t}$ is independent of w

In fact, these bulleted properties are really all that are necessary for much of our analysis, and so we could have assumed these properties directly, rather than assuming multiplicative separability. We make the stronger assumptions because they make the (already very complicated) proof considerably more transparent and less cumbersome and because the required property of the imputed effort function is hard to interpret and hard to verify except under the multiplicative separability assumptions we have made.

2.2 Information

We do assume that the firm knows the distribution of characteristics of workers so that the firm's optimization problem will be well-posed. We assume that the workers know their own characteristics (production and cost functions) and observe task types, but workers do not need to know characteristics or distributions of characteristics of other workers.

3 Matching

In each period workers are matched with tasks, choose effort, produce output and receive payment. Because only output (not effort or worker type) is observed, it seems natural to assume that the matching of workers to task depends only on past history of output. We focus on the simplest and most obvious matching rule: workers are matched to tasks according to the ranking of output produced in the (immediately) previous period. Recall that the spaces of workers and tasks are $W = [B, 1]$, $T = [B, 1]$ and that the total masses of workers and tasks are $1 - B$.

An *output mapping* is a (measurable) map $G : W \to [0, \infty)$; we interpret

$G(w)$ as the output produced by worker $w \in W$. An *output distribution* is a (measurable) mapping $\Gamma : [0, \infty) \to [0, 1 - B]$; we interpret $\Gamma(y)$ as the mass of workers who produce output at most y (so Γ is a cumulative distribution function). Given an output mapping G, the corresponding output distribution Γ is defined by setting $\Gamma(y)$ to be the (Lebesgue) measure of the set $\{w \in W : G(w) \leq y\}$. Notice that the output distribution Γ is unaffected by changing G on a set of measure 0 and in particular is independent of the output of any single worker.

Fix the current output distribution Γ and an output $y \in [0, \infty)$. If worker w produces output y in the current period then in the next period the *matching rule* μ assigns worker w to task $t = \mu(y) = B + \Gamma(y)$. That is, worker w is assigned to the task whose rank in the task distribution is precisely the same as the rank of y in the output distribution. In particular, if y is the worst output, then w will be assigned the worst task, and so forth.

This matching rule requires some comment. Consider an output map G and the corresponding output distribution Γ. As we have already noted, the distribution Γ is independent of the output of any single worker, so the output choice of worker w affects the task assigned to worker w but not the task assigned to any other worker. (Of course this is one reason we have chosen to work in a continuum model.) If G is not one-to-one then there will be workers $w, \hat{w}$ for which $G(w) = G(\hat{w})$, so workers $w, \hat{w}$ will be assigned the same task. Because we focus on equilibria in which the output mapping G *will be* one-to-one, we ignore this complication. However even if the actual output mapping G is one-to-one, it may be that worker w contemplates a deviation from G in which he/she produces output $y \neq G(w)$; in particular, worker w may contemplate producing the output $G(\hat{w})$ of some other worker $\hat{w}$. In that case, the matching rule would again assign workers $w, \hat{w}$ to the same task. This seems an unavoidable complication of the continuum model. (In a large finite model, worker w could simply produce an output slightly larger than $G(\hat{w})$ and this problem would not arise – but then worker w's choice would affect the task assigned to worker $\hat{w}$.) Because this will occur only as a counter-factual, we will ignore this potential complication as well.

3.1 Assortative Matching

We focus here on assortative matching because it is simple and also because it produces outcomes that are optimal for the firm, at least within the class of matching schemes that are constant over time and consistent with worker incentives. (See Section 7.)

4 The Workers' Subgame

In this Section we focus on the subgame played by the workers in the second stage. We therefore fix the matching rule described in Section 3 and a payment scheme $P(y) = \lambda y$ as described in Section 2. Given this matching rule and payment scheme, we define a notion of equilibrium in the subgame played by the workers; in the next Section we show that this equilibrium exists and is unique.

In each period, each worker w is assigned a task and chooses an effort level to exert. At the end of each period the entire output distribution is revealed.[8]

[8]Note that workers observe the output distribution but not necessarily the output mapping. In particular workers know how many other workers produced output below a

The history of worker w in period n is therefore the sequence of previous assignments of tasks, choices of effort and observed output distributions. In the current period, the worker is assigned a task and must choose an effort level, so a *(pure) strategy* for worker w is therefore a map $\sigma_w : \textit{history} \times T \to E$ from (past) history and (current) task to effort. Note that, given worker type w and task type t, effort e determines output $y = Y(e, w, t) = ep(w)q(t)$ and output determines effort $e = \Phi(y, w, t) = y/p(w)q(t)$ so there is no loss in viewing a strategy as a map from history and task to *output*.

Now fix a strictly increasing output mapping G and the corresponding output distribution Γ. Fix a worker w and a strategy σ_w for worker w. Suppose that other workers produce according to G in every period and in particular that the output distribution is Γ in every period. If worker w is initially assigned to the task t_0 in period 0, then the strategy σ_w determines the output y_0 to be produced in period 0, and the effort $e_0 = y_0/p(w)q(t_0)$ required to produce this output. The output level y_0 will place worker w at some point in the output distribution Γ and hence determine the task $t_1 = \mu(y_0)$ to which worker w will be assigned in period 1. The strategy σ_w determines the output y_1 to be produced in period 1 and the effort $e_1 = y_1/p(w)q(t_1)$ required, and so forth. Thus, assuming that other workers produce in such a way that the output distribution is Γ in every period, the strategy σ_w and the initial task assignment t_0 determine the entire history of worker w: in period n, worker w is assigned to task t_n and produces output y_n which requires effort $e_n = y_n/p(w)q(t_n)$. This yields worker w the utility $\lambda y_n - k(w)c(e_n)$ in

given level – but not the names of those workers.

period n and hence lifetime utility of

$$V_w(\sigma_w | t_0, \Gamma) = (1-\delta) \sum_{n=0}^{\infty} \delta^n [\lambda y_n - k(w) c(e_n)]$$

The strategy σ_w is *optimal from* t_0 (for worker w) if

$$V_w(\sigma_w | t_0, \Gamma) = \sup V_w(\widehat{\sigma}_w | t_0, \Gamma)$$

where the supremum is taken over all pure strategies $\widehat{\sigma}_w$.

A *steady state assortative equilibrium* (SSAE) for the workers subgame consists of a strictly increasing output map G and strategies σ_w for each worker such that

- $\sigma_w(h, w) = G(w)$ for every history h
- σ_w is optimal from $t_0 = w$

That is: When workers are matched assortatively each period, then workers produce output as prescribed by G and find it optimal to do so. Given this, the matching rule guarantees that, along the equilibrium path, matching is assortative in every period (worker w is matched with task $t = w$). As usual, we require optimality among *all* strategies, not just among constant or stationary strategies, so we allow for the possibility that worker w contemplates a strategy that calls for a complicated sequence of production plans, which would lead to a correspondingly complicated sequence of assignments to tasks.

Note that if other workers are behaving according to the prescription of steady state assortative equilibrium, then deviation by a single worker does not affect future matchings or future behavior of other workers. In particular, behavior along the equilibrium path in a steady state assortative equilibrium is completely determined by the output mapping G. We therefore identify a steady state assortative equilibrium with G itself; in particular, we ignore the out-of-equilibrium plans of workers. This should cause no confusion.

5 Existence and Uniqueness of SSAE

Our fundamental result is that, if $B > 0$ (so that even the worst worker when matched to the worst task can produce strictly positive output) then there exists a unique steady state assortative equilibrium. We defer the (complicated) proof to the Appendix.

Theorem 1 *Fix the payment schedule $\lambda \in (0,1)$. If $B > 0$ (so that even the worst worker matched to the worst task can produce strictly positive output) then:*

(i) there is a unique steady state assortative equilibrium

(ii) the equilibrium output mapping G is continuously differentiable and is the unique increasing solution to the ordinary differential equation

$$G'(w) = \delta \left(\frac{k(w)q'(w)c'\left(\frac{G(w)}{p(w)q(w)}\right)}{q(w)\left[k(w)c'\left(\frac{G(w)}{p(w)q(w)}\right) - \lambda p(w)q(w)\right]} \right) G(w) \qquad (1)$$

subject to the initial condition that the worst worker, matched to the worst task, chooses the level of output that maximizes current utility

(iii) worker utility is strictly increasing in worker type w.

5.1 Comments

We have assumed that all workers discount the future at the same rate δ, but this assumption is not necessary. We could allow worker-specific discount factors $\delta(w)$, provided that the map $\delta : W \to (0, 1)$ is smooth and satisfies natural assumptions; the only change would be that in the ordinary differential equation (1) the factor δ would become a factor $\delta(w)$.

The assumption that $B > 0$ means that the worst worker and worst task are not worthless, which seems natural. However, it is also of interest to treat the setting in which the worst worker and worst task are worthless, not least because – as we show in Section 8 – it is only in that setting that we can find solutions in closed form (for specific functional forms). Unfortunately, in the latter setting, the ODE's for the solution and for the inverse of the solution are no longer well-behaved (the right-hand sides are not Lipschitz), so existence and uniqueness of solutions are not guaranteed. In Section 8 we show how this problem can be addressed (at least in some settings).

As we have noted earlier, we do not require that workers know very much about the environment, and hence there is no reason to suppose that workers can solve for steady state assortative equilibrium. SSAE simply describes a particular state of the system; our knowledge of the parameters allows us to solve for this particular state of the system, but we do not offer any mechanism by which the system reaches this state.

6 Equilibrium in the Two-Stage Game

Having established existence and uniqueness of (steady-state assortative) equilibrium in the workers subgame, we can now show that the whole two-stage game admits an equilibrium. To be precise, by an "equilibrium" in the two stage game we mean

(i) for workers: for every $\lambda \in (0,1)$, workers play the unique steady-state assortative equilibrium G^{λ} for the workers subgame, given the payment scheme $P(y) = \lambda y$, and

(ii) for the firm: a particular choice $\lambda^* \in (0,1)$ which maximizes the firm's profit given that, for each λ workers follow the unique steady-state assortative equilibrium G^{λ}.

Evidently we are imposing a kind of subgame perfection.

Theorem 2 *The two-stage game admits an equilibrium.*

The proof, deferred to the Appendix, requires demonstrating that firm profit is continuous in λ and achieves a maximum for λ in the open interval $(0,1)$.

7 Profit Comparisons

It seems natural to suppose that the firm seeks to maximize (expected) profit: the (expected) output produced minus (expected) payments. We have required that the firm choose and commit in the first stage to a linear payment scheme and a particular matching rule and focused on assortative matching. But it is not entirely obvious that assortative matching is best for the firm. To see why, recall that our assumptions guarantee that output is supermodular in all variables (effort, worker type, task type). If output were independent of worker effort, this would be enough to guarantee that total output (and hence firm profit, which is a fixed fraction of total output) would be maximized by matching workers and tasks assortatively. However output is *not* independent of worker effort, and it might happen that the endogenous choice of worker effort undoes the optimality of assortative matching, in which case there would be no reason to suppose that the firm would wish to match workers and tasks assortatively and hence no reason to suppose that the firm would wish to use the matching rule on which we have focused. This is not merely a hypothetical issue; below we provide an example to show that if marginal cost were not log-concave in effort the firm might actually prefer *random matching* to assortative matching. However, under the assumptions we have actually made, the problem disappears: the endogenous choice of workers does not undo the optimality of assortative matching. This enables us to make some obvious comparisons of firm profit in our equilibrium with firm profit in natural benchmark scenarios.

For a given fixed payment schedule $P(y) = \lambda y$, we compare profit Π_{SSAE} when workers play the steady state assortative equilibrium G^{λ} in the second stage game against three natural benchmarks.

- Π_{random} is what the firm's (expected) profit would be if the firm committed to matching workers and tasks *randomly* in each period and paying according to the given payment schedule $P(y) = \lambda y$, and workers then chose effort optimally given these commitments.
- Π_{assort} is what the firm's profit would be if the firm could actually observe workers' characteristics, committed to matching workers and tasks *assortatively* in each period and paying according to the given payment schedule $P(y) = \lambda y$, and workers then chose effort optimally

given these commitments.

- Π_{FI} is what the firm's profit would be in the full-information setting, in which the firm could actually observe workers' characteristics and could use this information to match workers and tasks and to set a worker/task-specific payment schedule, and workers then chose effort optimally, given these assignments and payment schedules.

It must be kept in mind that, in our setting, the firm *cannot* observe workers' characteristics so only the first benchmark represents something that the firm could actually achieve. However comparing firm profit in the steady state assortative equilibrium with these benchmarks provides a useful measure of how much the firm is able to achieve with the limited power given to it in comparison with what it might achieve if it had greater power.

Theorem 3 *For a given fixed payment scheme $P(y) = \lambda y$, firm profits in the steady state assortative equilibrium and the benchmarks are ordered as follows:*

$$\Pi_{\mathrm{random}} < \Pi_{\mathrm{assort}} < \Pi_{\mathrm{SSAE}} < \Pi_{\mathrm{FI}}$$

(Note that the inequalities are strict.)

It is not hard to see that $\Pi_{\mathrm{assort}} < \Pi_{\mathrm{SSAE}}$ and that $\Pi_{\mathrm{SSAE}} < \Pi_{\mathrm{FI}}$. In order to show that $\Pi_{\mathrm{random}} < \Pi_{\mathrm{assort}}$, we show that when worker w is matched to task t and chooses effort optimally, the *imputed output* $Y^*(w,t)$ is supermodular in w, t; so the optimal matching is assortative. The details are in the Appendix.

7.1 Non-assortative Matching: An Example

We present here an example that satisfies all of our assumptions with the exception that marginal cost is not log-concave, and show that for this example, assortative matching is the least-efficient matching – rather than the most-efficient matching – and hence that firm profit in steady-state assortative equilibrium may be *less* than it would be (with the same payment scheme) if the firm simply committed to matching workers and tasks *randomly*.

Fix parameters $\alpha, \beta \in (0,1)$ and a payment coefficient $\lambda \in (0,1)$. Set $p(w) = w^\beta$, $q(t) = t$, $k(w) = \frac{1}{w}$, and $c(e) = 1 - (1-e)^\alpha$. For each w, t define $e^*(w,t)$ to be the effort choice of worker w if it is matched to task t and behaves myopically. That is, $e^*(w,t)$ is the effort choice that maximizes

$$U(e,w,t) = \lambda Y(e,w,t) - C(e,w) = \lambda w^\beta te - (1/w)\left[1-(1-e)^\alpha\right] \qquad (2)$$

Define *imputed output* by $Y^*(w,t) = Y(e^*(w,t),w,t)$; this is the output produced when worker w is matched to task t and chooses the myopically optimal level of effort.

We assert that if $B \geq \left(\frac{\alpha}{\lambda}\right)^{\frac{1}{3}}$ and β is small enough then $Y^*(w,t)$ is *submodular*; i.e., the mixed partial $\frac{\partial^2 Y^*}{\partial w \partial t} < 0$. This is a straightforward computation. If worker w is matched to task t and exerts effort e its utility is $U(e,w,t)$ as in (2). Because $B \geq \left(\frac{\alpha}{\lambda}\right)^{\frac{1}{3}}$, the myopically optimal effort will be

$$e^*(w,t) = 1 - \left(\frac{\alpha}{\lambda w^{1+\beta} t}\right)^{\frac{1}{1-\alpha}}.$$

Differentiating shows that

$$\frac{\partial^2 Y^*(w,t)}{\partial w \partial t} = \frac{\alpha\beta+1}{w^{\frac{1-\alpha+\alpha\beta}{1-\alpha}}}\left[\frac{\beta w^{\frac{\beta}{1-\alpha}}}{\alpha\beta+1} - \left(\frac{\alpha}{\lambda}\right)^{\frac{1}{1-\alpha}} \frac{\alpha}{(1-\alpha)^2}\left(\frac{w}{t}\right)^{\frac{1}{1-\alpha}}\right]$$

Since $w \leq 1$ and $\left(\frac{w}{t}\right)^{\frac{1}{1-\alpha}} \geq B^{\frac{1}{1-\alpha}}$, we have

$$\frac{\partial^2 Y^*(w,t)}{\partial w \partial t} \leq \frac{\alpha\beta+1}{w^{\frac{1-\alpha+\alpha\beta}{1-\alpha}}} \left[\frac{\beta}{\alpha\beta+1} - \left(\frac{\alpha}{\lambda}\right)^{\frac{1}{1-\alpha}} \frac{\alpha}{(1-\alpha)^2} B^{\frac{1}{1-\alpha}}\right]$$

Since $\left(\frac{\alpha}{\lambda}\right)^{\frac{1}{1-\alpha}} \frac{\alpha}{(1-\alpha)^2} B^{\frac{1}{1-\alpha}}$ is a positive constant, and since $\lim_{\beta\to 0} \frac{\beta}{\alpha\beta+1} = 0$, if $\beta > 0$ is small enough then

$$\frac{\beta}{\alpha\beta+1} < \left(\frac{\alpha}{\lambda}\right)^{\frac{1}{1-\alpha}} \frac{\alpha}{(1-\alpha)^2} B^{\frac{1}{1-\alpha}}.$$

Hence for any such β we see that $\frac{\partial^2 Y^*(w,t)}{\partial w \partial t} < 0$; i.e. Y^* is submodular.

Because Y^* is submodular, $-Y^*$ is supermodular. Hence if workers myopically choose effort to maximize current utility, total output is *minimized* when workers and tasks are matched assortatively. Because firm profit is a fixed fraction of total output, firm profit is also *minimized* when workers and tasks are matched assortatively. In particular, if $\delta = 0$, firm profit in the steady state assortative equilibrium is *strictly less* than it would be if the firm committed to matching workers and tasks *randomly* (but used the same payment schedule). It follows immediately that if $\delta > 0$ but sufficiently small then firm profit in the steady state assortative equilibrium is *strictly less* than it would be if the firm committed to matching workers and tasks *randomly* (but used the same payment schedule); that is $\Pi_{\text{random}} > \Pi_{\text{SSAE}}$.

8 Examples

To illustrate the general results of Theorems 1 and 3, it seems useful to consider examples for which solutions can be computed in closed form. Unfortunately, even for the simplest functional forms it seems impossible to solve the differential equation (1) in closed form when $B > 0$ (so that the initial condition is that the worst worker is matched to the worst task and exerts the myopically optimal effort). To get around this problem, we take $B = 0$ and consider a class of functional forms for which the worst worker and the worst task are worthless (i.e., no level of effort can produce positive output). In this setting, we can explicitly write down closed form solutions, prove that the solutions are unique (a fact that no longer follows from standard uniqueness results) and show that the solution when $B = 0$ approximates the solution when $B > 0$ but small.

For the remainder of this Section, we assume $B = 0$ so the worker space is $W_0 = [0, 1]$ and the task space is $T_0 = [0, 1]$. We begin with the simplest functional forms: $Y(e, w, t) = ewt$, $C(e, w) = e^2$, $P(y) = \lambda y$ for $\lambda \in (0, 1)$; the analysis for some more general functional forms (discussed below) is almost the same although the algebra is much messier. Note that the worst worker and the worst task are worthless. We assert that, for this setting and these functional forms, there is a unique steady state assortative equilibrium:

$$G_0(w) = \left(\frac{2\lambda}{4 - \delta}\right) w^4$$

Moreover, if for each $B > 0$ we denote by G_B the unique steady state assortative equilibrium guaranteed by Theorem 1 when we keep output, cost, payment rule the same but restrict the worker and task spaces to $W_B = [B, 1], T_B = [B, 1]$ then $G_B \to G_0$ and $G'_B \to G'_0$ uniformly on every interval $[B, 1]$ with $B > 0$.

To see that G_0 is the unique steady state assortative equilibrium, note first that the same argument as in the proof of Theorem 1 shows that every steady state assortative equilibrium G is smooth, strictly increasing and satisfies the ODE (1), and that every strictly increasing solution to the ODE (1) is a steady state assortative equilibrium. For the given functional forms, the ODE (1) reduces to

$$G'(w) = \frac{2\delta G(w)^2}{[2G(w) - \lambda w^4]\, w} \tag{3}$$

Direct computation shows that the function G_0 solves (3) and satisfies the initial condition $G_0(0) = 0$ so to show that G_0 is the unique steady state assortative equilibrium it remains only to show that it is the unique increasing solution to (3) satisfying the initial condition. To see this, suppose that $\widehat{G}_0$ were another solution satisfying the initial condition. The ODE (3) is Lipschitz away from the critical curve where the denominator $[2G(w) - \lambda w^4]\, w$ of the right hand side of (3) is zero, so the solutions $G_0, \widehat{G}_0$ cannot cross for $w \neq 0$. In particular, if $G_0(w) > \widehat{G}_0(w)$ for some w then $G_0(w) > \widehat{G}_0(w)$ for all w, and vice versa. However, it is easily checked that the right hand side of (3) is strictly decreasing in G, so if $G_0(w) > \widehat{G}_0(w)$ for all w it would necessarily be the case that $G_0'(w) < \widehat{G}_0'(w)$ for all w and vice versa, which would violate the Mean Value Theorem. Hence G_0 is the unique increasing solution to (3) satisfying the initial condition and hence the unique steady state assortative equilibrium.

To see that $G_B \to G_0$ and $G_B' \to G_0'$, note that, because solutions to the ODE (3) cannot cross it must be the case that the solutions G_B are strictly decreasing in B (i.e. $G_B(w) < G_{\hat{B}}(w)$ if $B > \hat{B}$ and $w \geq B$) and hence converge to *some* function F. The fact that the solutions G_B solve the ODE (3) guarantee that, away from the critical curve, the solutions G_B are equi-uniformly differentiable (i.e., the difference quotients converge to the derivative at a rate that is independent of $w \in [b, 1]$ provided that $0 < b < B$), and hence that the functions G_B and their derivatives G_B' converge uniformly to F and F' (respectively) for $w \in [b, 1]$. It follows that the limit function F satisfies the ODE (3) on $(0, 1]$ and that $\lim_{w \to 0} F(w) = 0$, and hence that $F = G_0$, so we obtain the desired convergence assertion.

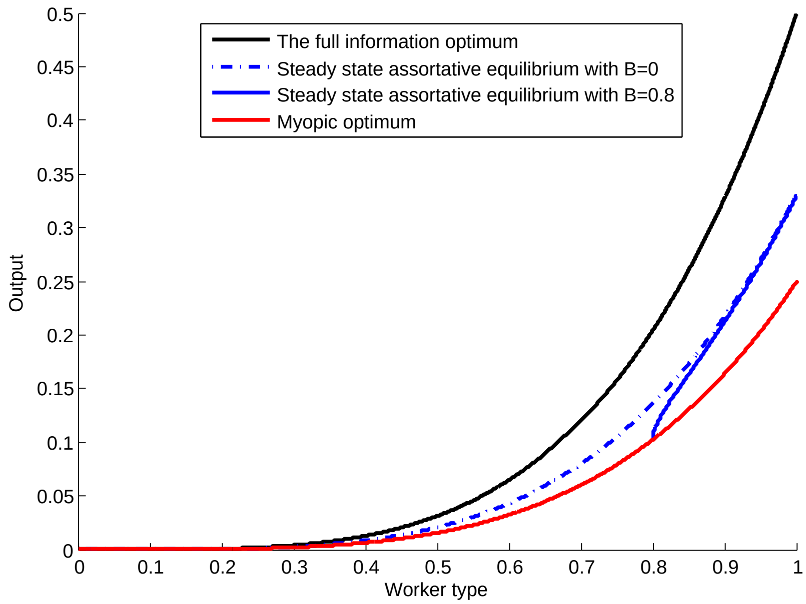

Figure 1: Solution in the Main Example

Figure 1 shows the solutions G_B and G_0 and their positions above the myopically optimal solution (i.e. the output produced when workers are matched assortatively each period and produce the myopically optimal level) and below the full-information solution (i.e. the output produced when workers are matched assortatively each period and the firm extracts the full surplus consistent with the incentives of workers).

We can now compute the firm profit for the steady state assortative equilibrium and the benchmarks. Fix $\lambda \in (0,1)$.

- If workers and tasks are always matched randomly and the payment schedule $P(y) = \lambda y$ is fixed, then worker w matched with task t will choose effort to maximize $\lambda ewt - e^2$ and hence will choose effort $e = \lambda wt/2$ and produce output $y = \lambda w^2 t^2/2$. Because the firm retains the fraction $1-\lambda$ of output, firm profit will be

$$\Pi_{\text{random}} = \int_0^1 \int_0^1 (1-\lambda)(\lambda w^2 t^2/2)\, dt\, dw = [(1-\lambda)\lambda][1/18]$$

- If workers and tasks are always matched assortatively and the payment schedule $P(y) = \lambda y$ is fixed, then worker w matched with task w will choose effort to maximize $\lambda ew^2 - e^2$ and hence will choose effort $e = \lambda w^2/2$ and produce output $y = \lambda w^4/2$. Firm profit will be

$$\Pi_{\text{assort}} = \int_0^1 (1-\lambda)[\lambda w^4/2]\, dw = [(1-\lambda)\lambda][1/10]$$

- In the steady state assortative equilibrium, worker w is matched with task $t = w$ and produces output $G_0(w) = [2\lambda/(4-\delta)]\, w^4$ so firm profit will be

$$\Pi_{\text{SSAE}} = \int_0^1 (1-\lambda)(2\lambda/(4-\delta)]w^4)\, dw = [(1-\lambda)\lambda][2/5(4-\delta)]$$

- Finally, we consider the full information firm optimum; i.e. the profit the firm would make if it knew the characteristics of each worker and could offer worker/task-specific payment schedules. In that case, the firm can extract the full surplus from each worker. If worker w were matched to task t the firm extracts the full surplus by offering a payment schedule to maximize profit (output net of payment) subject to the incentive constraint on worker effort. The firm would therefore induce the effort level e that maximizes $ewt - e^2$. This effort level is $e = wt/2$ so optimal profit would be $\pi(w,t) = w^2t^2/2 - w^2t^2/4 = w^2t^2/4$. Note that the function $\pi(w,t)$ is supermodular in w, t so the matching that yields optimal profit is assortative. Hence in the full-information firm optimum, worker w is matched to task $t = w$, produces output $w^4/2$ and receives wage $w^4/4$. Firm profit in the full-information optimum is

$$\Pi_{\mathrm{FI}} = \int_0^1 \left(w^4/4\right)\, dw = 1/20$$

Because $\delta \in (0,1)$ we see that $1/10 < 2/5(4-\delta) < 2/15$; because $\lambda \in (0,1)$, we see that $(1-\lambda)\lambda \leq 1/4$ so

$$\Pi_{\mathrm{random}} < \Pi_{\mathrm{assort}} < \Pi_{\mathrm{SSAE}} < \Pi_{\mathrm{FI}}$$

Evidently, $\Pi_{\mathrm{random}}, \Pi_{\mathrm{random}}, \Pi_{\mathrm{SSAE}}$ are all maximized by taking $\lambda = 1/2$; noting that $\lim_{\delta\to 1} 2/5(4-\delta) = 2/15$ we see that if the firm chooses the optimal (linear) payment schedule and workers are perfectly patient then we have

$$\begin{array}{ccccccc}
\Pi_{\mathrm{random}} & < & \Pi_{\mathrm{assort}} & < & \Pi_{\mathrm{SSAE}} & < & \Pi_{\mathrm{FI}} \\
\| & & \| & & \| & & \| \\
\frac{1}{72} & < & \frac{1}{40} & < & \frac{1}{30} & < & \frac{1}{20}
\end{array}$$

A similar analysis can be carried through for the wide class of functional forms $Y(e, w, t) = ew^a t^b$, $C(e, w) = e^d w^{-s}$, $P(y) = \lambda y$ (assuming $a, b > 0$, $a + b \geq 1$, $d \geq 2$, $s \geq 0$). The unique steady state assortative equilibrium has the form $G_0(w) = Aw^\gamma$, where

$$\begin{aligned} \gamma &= \frac{(a+b)d - s}{d-1} \\ A &= \left(\frac{\lambda\gamma}{d\gamma - \delta b} \right)^{1/(d-1)} \end{aligned}$$

Solving for the profit in the first two benchmarks and in the steady state assortative equilibrium shows that each of $\Pi_{\text{random}}, \Pi_{\text{assort}}, \Pi_{\text{SSAE}}$ is the product of $(1-\lambda)\lambda^d$ and terms that involve only the exponents a, b, d, s and the discount factor δ but do not involve λ. (We leave the straightforward but messy algebra to the reader.) Hence for each of these benchmarks, the optimal (linear) payment schedule maximizes $(1-\lambda)\lambda^d$, which is accomplished by taking $\lambda = d/(1+d)$. Thus, at least for this class of functional forms, the optimal (linear) payment schedule depends only on the common cost factor $c(e) = e^d$ and not on the the productivity $p(w) = w^a$ of workers, the quality of tasks $q(t) = t^b$ or the worker-specific cost factor $k(w) = w^{-s}$.

8.1 Comment

We noted in Subsection 5.1 that existence and uniqueness of solutions to the ODE – and hence existence and uniqueness of SSAE – presents additional complictions when the worst worker and worst task are worthless. The examples above address these complications by demonstrating that solutions G_B for the restricted domains $W_B = [B, 1], T_B = [B, 1]$ converge to solutions G_0 on the domains $W_0 = [0, 1], T_0 = [0, 1]$. This argument is perfectly general, so that existence of SSAE is guaranteed even when the worst worker and worst task are worthless. Uniqueness seems more complicated however. In the examples, uniqueness is derived from the fact that the right-hand side of the ODE is decreasing in G. Unfortunately, this fact does not hold for more general functional forms. A sufficient condition is that the common cost factor satisfies $c'(e) \leq c(e)/e + c(e)c''(e)/c'(e)$. The reader can easily verify that this inequality is satisfied when $c(e) = e^{\alpha}$ for $\alpha > 1$, as in the examples, but it does not follow from our other assumptions and does not seem to have any obvious intuitive interpretation.

9 Conclusion

In this paper we have formulated and analyzed a dynamic assignment model with one-sided adverse selection (unobserved worker characteristics) and moral hazard (unobserved worker effort). For this environment we have defined a notion of steady state equilibrium in which workers are matched to tasks endogenously on the basis of observable output and shown that (for a given payment schedule) such an equilibrium exists and is unique. At equilibrium, adverse selection is eliminated and moral hazard is mitigated. Firm profit in equilibrium is compared with natural benchmarks. For specific examples, we find closed-form solutions and solve for the optimal (linear) payment scheme.

In the environment we have considered here, a single firm owns a large family of tasks and outsources them to a large family of workers in each period. In this environment, we have assumed that the single firm commits to a payment schedule that depends on output and matches tasks to workers. Equilibrium is driven by competition among workers.

A natural extension of this environment would consider a family of firms each of which owns a single task which it seeks to have performed by a single worker each period. In this extension, it would be natural to assume that each firm sets a payment schedule that depends on output and task type, and that some central agency/platform matches tasks to workers. In this environment, it seems natural to view the payment schedules set by different firms as part of the equilibrium, and determined in equilibrium by competition among firms. Gretsky, Ostroy and Zame (1999) provides some hints as to how this might occur.

Another extension that seems natural would consider not workers and tasks but workers of different kinds with complementary skills, so that the issue is matching complementary workers in teams. In that environment, it seems natural to contemplate adverse selection and moral hazard on both sides, which would make analysis very challenging indeed.

Appendix

Proof of Theorem 1 The proof of Theorem 1 is in two parts. In the first part we introduce the (weak) notion of imitative equilibrium and use results of Mailath (1987) to show that there is a unique imitative equilibrium, that the imitative equilibrium satisfies the ODE (1), and that in the imitative equilibrium, worker utility is strictly increasing in worker type. In the second – and much more complicated – part, we show that the unique imitative equilibrium is in fact a steady state assortative equilibrium.

To define an imitative equilibrium, fix a strictly increasing output function G that satisfies the initial condition that the worst worker, matched with the worst task, produces the myopically optimal output. For each worker w, task τ and output level y define $V(w, \tau, y)$ to be the long-run utility of worker w when it is matched to task w in period 0 and to task τ in every succeeding period and produces output y in every period. Keeping in mind that the effort required to produce output y depends on the task, we see that

$$\begin{aligned} V(w, \tau, y) &= (1-\delta)\left[P(y) - C\left(\Phi(y, w, w), w\right)\right] \\ &\quad + \delta\left[P(y) - C\left(\Phi(y, w, \tau), w\right)\right] \\ &= (1-\delta)\left[\lambda y - k(w)\, c\left(\frac{y}{p(w)q(w)}\right)\right] \\ &\quad + \delta\left[\lambda y - k(w) c\left(\frac{y}{p(w)q(\tau)}\right)\right] \end{aligned}$$

If $y \in G([B, 1])$ then $y = G(\hat{w})$ is the output specified for worker $\hat{w} = G^{-1}(y)$. Because G is strictly increasing, the matching rule matches a worker producing output $G(\hat{w})$ to the task $\hat{w}$. If worker w produces output y in every period then in period 1 and in every succeeding period worker w will be matched to task $\hat{w} = G^{-1}(y)$. Thus, in following this strategy, worker w is imitating worker $\hat{w}$.

We say G is an *imitative equilibrium* if no worker can gain by imitating another worker. That is, for every $w \in W$ we have the equivalent conditions

$$\begin{aligned} G(w) &\in \operatorname{argmax}_{y \in G([B,1])} V\left(w, G^{-1}(y), y\right) \\ w &\in \operatorname{argmax}_{\hat{w} \in W} V\left(w, \hat{w}, G(\hat{w})\right) \end{aligned}$$

Our first task is to prove that a unique imitative equilibrium equilibrium exists and establish some of its properties. The key is that when we restrict to imitation strategies, we have turned the original infinite horizon game into a signaling game to which the results of Mailath can be applied once we verify the necessary properties of the value function V. We formalize all of this as a Lemma.

Lemma 1 *There is a unique imitative equilibrium G. It is strictly increasing, continuously differentiable, and satisfies the ODE (1) with the specified initial condition, and worker utility is strictly increasing in worker index.*

Proof We first show that the function V satisfies Mailath's conditions.

Condition (1): "smoothness" Based on our assumptions, the functions P, C, Φ are all twice continuously differentiable. As a result, $V(w, \tau, y)$ is twice differentiable on $[B, 1]^2 \times \mathbb{R}$.

Condition (2): "belief monotonicity" Differentiating V with respect to τ yields

$$\frac{\partial V(w, \tau, y)}{\partial \tau} = \delta \left[\frac{y k(w) q'(\tau)}{p(w) \left[q(\tau)\right]^2} \right] c' \left(\frac{y}{p(w) q(\tau)} \right) > 0$$

for all $w \in W$, $\tau \in T$ and $y > 0$. This is Condition (2).

Condition (3): "type monotonicity" Differentiating V with respect to w and then with respect to y yields

$$\begin{aligned}\frac{\partial^2 V(w,\tau,y)}{\partial w \partial y} &= (1-\delta)\left[\frac{-k'(w)}{p(w)q(w)} + \frac{k(w)p'(w)}{[p(w)]^2\, q(w)} + \frac{k(w)q'(w)}{p(w)\,[q(w)]^2}\right] c'\left(\frac{y}{p(w)q(w)}\right) \\ &+ (1-\delta)\left(\frac{yk(w)}{[p(w)q(w)]^3}\right)[p'(w)q(w) + p(w)q'(w)]\, c''\left(\frac{y}{p(w)q(\tau)}\right) \\ &+ \delta\left[\frac{-k'(w)p(w) + k(w)p'(w)}{[p(w)]^2\, q(\tau)}\right] c'\left(\frac{y}{p(w)q(\tau)}\right) \\ &+ \delta\left[\frac{yk(w)p'(w)}{[p(w)q(\tau)]^3}\right] c''\left(\frac{y}{p(w)q(\tau)}\right)\end{aligned}$$

Keeping in mind that k is weakly decreasing, so that $k' \leq 0$, we see that each of the terms on the right-hand side are positive, so $\frac{\partial^2 V(w,\tau,y)}{\partial w \partial y} > 0$ for all $w \in W$, $\tau \in T$ and $y > 0$. This is Condition (3).

Condition (4) Recalling that the function c is strictly convex, we have

$$\frac{\partial^2 V(w,w,y)}{\partial y^2} = -\left(\frac{k(w)}{[p(w)q(w)]^2}\right) c''\left(\frac{y}{p(w)q(w)}\right) < 0$$

for all $w \in W$, $\tau \in T$ and $y > 0$. Thus, $V(w,w,y)$ is strictly concave in y.

Moreover, we have

$$\frac{\partial V(w,w,y)}{\partial y} = \lambda - \left[\frac{k(w)}{p(w)q(w)}\right] c'\left(\frac{y}{p(w)q(w)}\right)$$

The right-hand side is strictly positive at $y = 0$ and is strictly decreasing in y, so for each $w \in W$ the equation $\frac{\partial V(w,w,y)}{\partial y} = 0$ has a unique solution in y, and the unique solution maximizes $V(w,w,y)$. Together, these are Condition (4).

Condition (5): "boundedness" We have already noted that $V(w,w,y)$ is strictly concave in y so there does not exist any $w \in W$ and $y \geq 0$ such that $\frac{\partial^2 V(w,w,y)}{\partial y^2} \geq 0$. Hence, Condition (5) is satisfied.

Since $V(w,\tau,y)$ is increasing in τ, Mailath's initial condition is that $G(B) = \arg\max_y V(B,B,y)$, namely the worst worker B, when matched to the worst task B, chooses myopically optimal output; this is of course precisely our initial condition.

We have verified Mailath's Conditions (1)-(5) and we are restricting to an output function G that satisfies the initial condition so Mailath's Theorems 1 and 2 guarantee that every imitative equilibrium G is continuous on $[B,1]$, smooth and strictly monotonic on $(B,1)$ and solves the ODE

$$G'(w) = -\frac{\left.\frac{\partial V(w,\tau,y)}{\partial \tau}\right|_{\tau=w,y=G(w)}}{\left.\frac{\partial V(w,w,y)}{\partial y}\right|_{y=G(w)}}$$

Straightforward calculation shows that this ODE coincides with (1).

Mailath's Theorem 2 shows that $G'(w)$ has the same sign as $\frac{\partial V(w,\tau,y)}{\partial w \partial y}$, which we have already shown to be positive; hence every imitative equilibrium is strictly increasing. Moreover, because

$$\left.\frac{\partial V(w,\tau,y)}{\partial \tau}\right|_{\tau=w} = \delta y \frac{k(w)q'(w)}{p(w)\left[q(w)\right]^2} c'\left(\frac{y}{p(w)q(w)}\right)$$

which is bounded for each $w \in W$, $y \in [0,\infty)$ so the ODE has a unique solution that satisfies the initial condition.

We can therefore conclude that there is a unique imitative equilibrium G, that G is smooth and strictly increasing, and that G is the unique solution to the ODE (1) with the initial condition that $G(B)$ is worker B's myopically optimal output when matched to task B.

Finally, to show that worker utility is increasing in worker type note that in an imitative equilibrium, worker w receives the same payoff in each period so its long-run utility is $U(w) = \lambda G(w) - k(w)c\left(\frac{G(w)}{p(w)q(w)}\right)$. Differentiating and doing the requisite algebra yields

$$\begin{aligned}
U'(w) &= \lambda G'(w) - k'(w)c\left(\frac{G(w)}{p(w)q(w)}\right) \\
&\quad - k(w)c'\left(\frac{G(w)}{p(w)q(w)}\right)\left[\frac{G'(w)}{p(w)q(w)} - \frac{G(w)p'(w)}{[p(w)]^2 q(w)} - \frac{G(w)q'(w)}{p(w)[q(w)]^2}\right] \\
&= \left[\lambda - \frac{k(w)c'\left(\frac{G(w)}{p(w)q(w)}\right)}{p(w)q(w)}\right] G'(w) - k'(w)c\left(\frac{G(w)}{p(w)q(w)}\right) \\
&\quad + k(w)c'\left(\frac{G(w)}{p(w)q(w)}\right) G(w)\left[\frac{p'(w)}{[p(w)]^2 q(w)} + \frac{q'(w)}{p(w)[q(w)]^2}\right] \\
&= -\delta k(w)c'\left(\frac{G(w)}{p(w)q(w)}\right) G(w)\frac{q'(w)}{p(w)[q(w)]^2} - k'(w)c\left(\frac{G(w)}{p(w)q(w)}\right) \\
&\quad + k(w)c'\left(\frac{G(w)}{p(w)q(w)}\right) G(w)\left[\frac{p'(w)}{[p(w)]^2 q(w)} + \frac{q'(w)}{p(w)[q(w)]^2}\right] \\
&= k(w)c'\left(\frac{G(w)}{p(w)q(w)}\right) G(w)\left[\frac{p'(w)}{[p(w)]^2 q(w)} + \frac{(1-\delta)q'(w)}{p(w)[q(w)]^2}\right] \\
&\quad - k'(w)c\left(\frac{G(w)}{p(w)q(w)}\right)
\end{aligned}$$

Since k is weakly decreasing, this last expression is strictly positive, so worker utility is increasing in worker type w. This completes the proof. ■

Lemma 1 constitutes the first part of the proof of Theorem 1. We now turn to the second part, showing that an imitative equilibrium is a steady state assortative equilibrium. We arrange the proof as a sequence of lemmas.

By definition, an imitative equilibrium has the property that no worker has a profitable deviation that consists of imitating some other worker; we must show that no worker has any profitable deviation at all. We first show that if a profitable deviation exists then there is a profitable finite deviation. This is a simple and familiar consequence of discounting so we omit the proof.

Lemma 2 *Let G be the unique imitative equilibrium. If worker w has a profitable deviation from G then worker w has a profitable deviation from G in which, after a finite number of periods, it produces output $G(w)$ forever.*

We now study optimal finite-period deviations. Because G is continuous, the range of G is precisely the interval $[G(B), G(1)]$. First, we show that it is dominated for workers to choose output outside the interval $[G(B), G(1)]$ in any period.

Lemma 3 *Choosing output $y \notin [G(B), G(1)]$ is dominated in every period.*

Proof First consider the case in which the worker contemplates producing output less than $G(B)$. In view of the initial condition, $G(B)$ is worker B's myopically optimal output when matched to task B. We have shown that $G(B)$ is the solution to the equation

$$\frac{\partial V(B, B, y)}{\partial y} = 0.$$

Hence

$$\lambda - \left[\frac{k(B)}{p(B)q(B)}\right]\left[c'\left(\frac{G(B)}{p(B)q(B)}\right)\right] = 0.$$

Moreover, for every $w \in [B, 1]$, $\tau \in [B, 1]$, and $y < G(B)$ we have

$$\begin{aligned}
\frac{\partial\left[\lambda y - c\left(y/p(w)q(\tau)\right)k(w)\right]}{\partial y} &= \lambda - \frac{k(w)}{p(w)q(\tau)}c'\left(\frac{y}{p(w)q(\tau)}\right) \\
&> \lambda - \frac{k(B)}{p(B)q(B)}c'\left(\frac{G(B)}{p(B)q(B)}\right) \\
&= 0.
\end{aligned}$$

That is, worker w's current payoff is strictly increasing in its output at $y < G(B)$. Hence, if in period N, worker w produces output $y < G(B)$ then its current payoff will be less than if it produced output $G(B)$. Moreover, since it will be producing the worst output of any worker, it will be matched in period $N+1$ to the worst task (i.e. task B) – exactly as if it had produced output $G(B)$. Hence producing output $y < G(B)$ in period N is dominated by producing output $G(B)$.

Now consider the case in which the worker contemplates producing output greater than $G(1)$. Since $G(w)$ is strictly increasing, and since the numerator of the ODE (1) is positive, the denominator of the ODE must also be positive, which is equivalent to

$$k(w)c'\left(\frac{G(w)}{p(w)q(w)}\right) > \lambda p(w)q(w),\ \forall w \in [B,1].$$

If worker $w \in [B,1]$ is matched to task $\tau \in [B,1]$ then for every $y > G(1)$, we have

$$\begin{aligned}\frac{\partial\left[\lambda y - c\left(y/p(w)q(\tau)\right)k(w)\right]}{\partial y} &= \lambda - \frac{k(w)}{p(w)q(\tau)}c'\left(\frac{y}{p(w)q(\tau)}\right) \\ &< \lambda - \frac{k(1)}{p(1)q(1)}c'\left(\frac{G(1)}{p(1)q(1)}\right)\end{aligned}$$

The last expression is negative, so we see that worker w's current payoff is strictly decreasing in its output at $y > G(1)$. Hence if in period N worker w produces output $y > G(1)$ then its current payoff will be less than if it produced output $G(1)$. Moreover, since it will be producing the best output of any worker, it will be matched in period $N+1$ to the best task (i.e. task 1) – exactly as if it had produced output $G(1)$. Hence producing output $y > G(1)$ in period N is dominated by producing output $G(1)$.[9] ■

[9] It is worth noting that there is nothing special about the output $G(1)$ in the sense

In view of the preceding lemmas, we can focus on finite deviations in which the worker w produces output in $[G(B), G(1)]$ in every period and returns to producing output $G(w)$ from some point on. Each such finite deviation is characterized by a sequence of periods $0, 1, \ldots, N$ in which the worker is matched with tasks $w_0 = w, w_1, \ldots, w_N$ and produces outputs $y_0 = G(w_1), y_1 = G(w_2), \ldots, y_N = G(w_{N+1})$.. (Note that these are precisely the outputs required in order to be matched to the specified tasks.) For lack of a better term, call this an N-*deviation*. From period $N+1$ on the worker will return to producing $G(w)$ forever. Given $w_1, \ldots w_N, w_{N+1}$ define

$$S_N(w_1, \ldots, w_N; w_{N+1}) \triangleq \sum_{n=0}^{N} \delta^t \left\{ \lambda G(w_{n+1}) - c \left[\frac{G(w_{n+1})}{p(w)q(w_n)} \right] k(w) \right\} \qquad (4)$$

This is the (discounted) utility worker w will obtain over this span of periods if the worker follows the given N-deviation.

Given w_{N+1} we study N-deviations $\{w_1, \ldots, w_N, w_{N+1}\}$. Among these, we search for those that are optimal, in the sense of maximizing S_N. To facilitate this search we first study how S_N depends on each of the intermediates $w_1, \ldots, w_N$. We then use that information to show that optimal N-deviations are strictly increasing or strictly increasing or constant. Finally we rule out the first two possibilities, and conclude that optimal N-deviations are constant From this it will follow quickly that no finite deviation from G can be profitable and hence that G is a SSAE, as asserted.

To see how S_N depends on w_n note first that w_n affects two terms in

that, for worker w, it is always the case that producing output $y > G(w)$ is worse in the current period than producing output y. However, if $y < G(1)$ then producing output $y > G(w)$ in the current period will result in a better task next period and so might be part of a profitable deviation.

the sum for S_N: It directly enters the output and cost in period $n-1$ and indirectly enters the payment and cost in period n by affecting the task matched to worker w in period n. More specifically, w_n enters only the following terms in the sum payoff S_N:

$$\begin{aligned} &\delta^{n-1}\left\{P\left[G(w_n)\right]-C\left[\Phi\left(G(w_n),w,w_{n-1}\right),w\right]\right\} \\ &\quad+\delta^{n}\left\{P\left[G(w_{n+1})\right]-C\left[\Phi\left(G(w_{n+1}),w,w_n\right),w\right]\right\}. \end{aligned} \tag{5}$$

Note that the only variables that appear in this expression are w_n, w_{n-1}, w_{n+1}. Define $Q_w(w_n; w_{n-1}, w_{n+1})$ to be the partial derivative of this expression (5) with respect to w_n. We can calculate it as

$$\begin{aligned} Q_w(w_n;w_{n-1},w_{n+1}) &= \delta^{n-1}\left\{\lambda-\left[\frac{k(w)}{p(w)q(w_{n-1})}\right]c'\left(\frac{G(w_n)}{p(w)q(w_{n-1})}\right)\right\}G'(w_n) \\ &\quad+\delta^{n}\left[\frac{k(w)q'(w_n)G(w_{n+1})}{p(w)\left[q(w_n)\right]^2}\right]c'\left(\frac{G(w_{n+1})}{p(w)q(w_n)}\right) \end{aligned}$$

We need to analyze the sign of Q_w; to do this it is convenient to do some preliminary work. Define

$$\begin{aligned} D_w(w_n,w_{n-1}) &= \lambda-\left[\frac{k(w)}{p(w)q(w_{n-1})}\right]c'\left(\frac{G(w_n)}{p(w)q(w_{n-1})}\right) \\ N_w(w_n,w_{n+1}) &= -\left[\frac{k(w)q'(w_n)G(w_{n+1})}{p(w)\left[q(w_n)\right]^2}\right]c'\left(\frac{G(w_{n+1})}{p(w)q(w_n)}\right) \end{aligned}$$

so that $Q_w = \delta^{n-1}D_wG'(w_n) - \delta^n N_w$. The facts we need about D_w, N_w are summarized in the following lemma.

Lemma 4 *For any w, w_{n-1}, w_n, and w_{n+1}, we have*

(i) $\frac{\delta N_w(w;w)}{D_w(w;w)} = G'(w)$.

(ii) $D_w(w_n; w_{n-1})$ *is strictly increasing in* w*, and strictly increasing in* w_{n-1}*;*

(iii) $N_w(w_n; w_{n+1})$ *is strictly decreasing in* w_{n+1}.

(iv) $N_w(w_n; w_{n+1}) < 0$ *for any* w, w_n, w_{n+1}, *and* $D_w(w; w) < 0$.

(v) Suppose that $w_{n-1} = w_n = w_{n+1} = \hat{w}$. *Then* $\frac{N_w(\hat{w};\hat{w})}{D_w(\hat{w};\hat{w})}$ *is strictly increasing in* w *in the domain of* w *such that* $D_w(\hat{w}; \hat{w}) < 0$.

Proof (i) follows directly from the ODE (1) and the definitions of D_w, N_w.

To obtain (ii), note that since $q(w_{n-1})$ is increasing in w_{n-1}, and $c'(e)$ is strictly increasing in e, we have $D_w(w_n; w_{n-1})$ is strictly increasing in w_{n-1}. Since $k(w)$ is decreasing in w, $p(w)$ is increasing in w, and $c'(e)$ is strictly increasing in e, we see that $D_w(w_n; w_{n-1})$ is strictly increasing in w.

To obtain (iii), note that since $G(w_{n+1})$ is increasing in w_{n+1} and $c'(e)$ is strictly increasing in e, so $N_w(w_n; w_{n+1})$ is strictly decreasing in w_{n+1}.

To obtain (iv), note that $G(w_{n+1}) \geq G(B) > 0$ for $B > 0$. Combined with our assumptions, this implies that $N_w(w_n; w_{n+1}) < 0$. Since $\frac{\delta N_w(w;w)}{D_w(w;w)} = G'(w)$ and $G'(w) > 0$, we have $D_w(w; w) < 0$.

To obtain (v), it is convenient to study $\frac{D_w(\hat{w};\hat{w})}{N_w(\hat{w};\hat{w})}$, instead of studying $\frac{N_w(\hat{w};\hat{w})}{D_w(\hat{w};\hat{w})}$ directly. We have

$$\begin{aligned}\frac{D_w(\hat{w};\hat{w})}{N_w(\hat{w};\hat{w})} &= \frac{\lambda - \frac{k(w)}{p(w)q(\hat{w})}c'\left(\frac{G(\hat{w})}{p(w)q(\hat{w})}\right)}{-\frac{k(w)q'(\hat{w})G(\hat{w})}{p(w)[q(\hat{w})]^2}c'\left(\frac{G(\hat{w})}{p(w)q(\hat{w})}\right)} \\ &= \underbrace{\frac{\lambda}{-\frac{k(w)q'(\hat{w})G(\hat{w})}{p(w)[q(\hat{w})]^2}c'\left(\frac{G(\hat{w})}{p(w)q(\hat{w})}\right)}}_{=A_1(w,\hat{w})} + \underbrace{\frac{q(\hat{w})}{q'(\hat{w})G(\hat{w})}}_{=A_2(w,\hat{w})}\end{aligned}$$

We first look at the first term $A_1(w, \hat{w})$. Note that the numerator of $A_1(w, \hat{w})$ is positive and independent of w, and that the denominator of

$A_1(w, \hat{w})$ is $N_w(\hat{w}; \hat{w})$, which is strictly increasing in w. Therefore, $A_1(w, \hat{w})$ is strictly decreasing in w.

We then look at the second term $A_2(w, \hat{w})$. It is clear that $A_2(w, \hat{w})$ is independent of w.

In summary, we have shown that $\frac{D_w(\hat{w};\hat{w})}{N_w(\hat{w};\hat{w})}$ is strictly decreasing in w. Note that $\frac{D_w(\hat{w};\hat{w})}{N_w(\hat{w};\hat{w})}$ can be positive or negative, and that $N_w(\hat{w}; \hat{w})$ is always negative. Hence, $\frac{N_w(\hat{w};\hat{w})}{D_w(\hat{w};\hat{w})}$ is strictly increasing in w in the domain of w such that $D_w(\hat{w}; \hat{w})$ does not change its sign. This completes the proof. ∎

The next lemma isolates the relevant properties of $Q_w(w_n; w_{n-1}, w_{n+1})$.

Lemma 5 *The sign of the derivative $Q_w(w_n; w_{n-1}, w_{n+1})$ satisfies:*

(i) When $w < w_{n+1}$, we have

$$Q_w(w_n; w_{n-1}, w_{n+1}) \begin{cases} > 0, & \text{if } w_n \leq \min\{w, w_{n-1}\} \\ < 0, & \text{if } w_n \geq \max\{w_{n+1}, w_{n-1}\} \end{cases}.$$

(ii) When $w > w_{n+1}$, we have

$$Q_w(w_n; w_{n-1}, w_{n+1}) \begin{cases} > 0, & \text{if } w_n \leq \min\{w_{n+1}, w_{n-1}\} \\ < 0, & \text{if } w_n \geq \max\{w, w_{n-1}\} \end{cases}.$$

(iii) When $w = w_{n+1}$, we have

$$Q_w(w_n; w_{n-1}, w_{n+1}) \begin{cases} > 0, & \text{if } w_n \leq \min\{w_{n+1}, w_{n-1}\} \text{ and } w_n < w_{n+1} \\ < 0, & \text{if } w_n \geq \max\{w, w_{n-1}\} \text{ and } w_n > w \end{cases},$$

Proof An immediate result of (i) of Lemma 4 is that

$$\begin{aligned} Q_w(w_n; w_{n-1}, w_{n+1}) &= \delta^{n-1} \cdot D_w(w_n; w_{N-1}) \cdot \left[\delta \cdot \frac{N_{w_n}(w_n; w_n)}{D_{w_n}(w_n; w_n)}\right] - \delta^t \cdot N_w(w_n; w_{n+1}) \\ &= \delta^n \cdot \left[D_w(w_n; w_{n-1}) \cdot \frac{N_{w_n}(w_n; w_n)}{D_{w_n}(w_n; w_n)} - N_w(w_n; w_{n+1})\right]. \end{aligned}$$

From this equality and the various parts of Lemma 4, we establish Lemma 5 case by case.

(i) Assume $w < w_{n+1}$. Suppose that $w_n \le \min\{w, w_{n-1}\}$. If $D_w(w_n; w_{n-1}) \ge 0$, since $G(w)$ is increasing and $N_w(w_n; w_{n+1})$ is always negative, we will have $Q_w(w_n; w_{n-1}, w_{n+1}) > 0$. If $D_w(w_n; w_{n-1}) < 0$, we will have $D_{w_n}(w_n; w_n) \le D_w(w_n; w_n) \le D_w(w_n; w_{n-1}) < 0$, and hence,

$$\begin{aligned}
Q_w(w_n; w_{n-1}, w_{n+1}) &= \delta^n \cdot \left[D_w(w_n; w_{n-1}) \cdot \frac{N_{w_n}(w_n; w_n)}{D_{w_n}(w_n; w_n)} - N_w(w_n; w_{n+1}) \right] \\
&\ge \delta^n \cdot \left[D_w(w_n; w_n) \cdot \frac{N_{w_n}(w_n; w_n)}{D_{w_n}(w_n; w_n)} - N_w(w_n; w_{n+1}) \right] \\
&\ge \delta^n \cdot \left[D_w(w_n; w_n) \cdot \frac{N_w(w_n; w_n)}{D_w(w_n; w_n)} - N_w(w_n; w_{n+1}) \right] \\
&> \delta^n \cdot \left[D_w(w_n; w_n) \cdot \frac{N_w(w_n; w_n)}{D_w(w_n; w_n)} - N_w(w_n; w_n) \right] \\
&= 0
\end{aligned}$$

In summary, we have $Q_w(w_n; w_{n-1}, w_{n+1}) > 0$ when $w_n \le \min\{w, w_{n-1}\}$.

Now suppose that $w_n \ge \max\{w_{n+1}, w_{n-1}\}$. Then we have $D_w(w_n; w_{n-1}) \le D_w(w_n; w_n) \le D_{w_n}(w_n; w_n) < 0$. Hence, we have

$$\begin{aligned}
Q_w(w_n; w_{n-1}, w_{n+1}) &= \delta^t \cdot \left[D_w(w_n; w_{n-1}) \cdot \frac{N_{w_n}(w_n; w_n)}{D_{w_n}(w_n; w_n)} - N_w(w_n; w_{n+1}) \right] \\
&\le \delta^t \cdot \left[D_w(w_n; w_n) \cdot \frac{N_{w_n}(w_n; w_n)}{D_{w_n}(w_n; w_n)} - N_w(w_n; w_{n+1}) \right] \\
&< \delta^t \cdot \left[D_w(w_n; w_n) \cdot \frac{N_w(w_n; w_n)}{D_w(w_n; w_n)} - N_w(w_n; w_{n+1}) \right] \\
&\le \delta^t \cdot \left[D_w(w_n; w_n) \cdot \frac{N_w(w_n; w_n)}{D_w(w_n; w_n)} - N_w(w_n; w_n) \right] \\
&= 0
\end{aligned}$$

This completes the proof of (i).

(ii) Assume $w > w_{n+1}$. Suppose that $w_n \leq \min\{w_{n+1}, w_{n-1}\}$. If $D_w(w_n; w_{n-1}) \geq 0$, we will have $Q_w(w_n; w_{n-1}, w_{n+1}) > 0$. If $D_w(w_n; w_{n-1}) < 0$, we will have $D_{w_n}(w_n; w_n) \leq D_w(w_n; w_n) \leq D_w(w_n; w_{n-1}) < 0$, and hence,

$$\begin{aligned}
Q_w(w_n; w_{n-1}, w_{n+1}) &= \delta^t \cdot \left[D_w(w_n; w_{n-1}) \cdot \frac{N_{w_n}(w_n; w_n)}{D_{w_n}(w_n; w_n)} - N_w(w_n; w_{n+1}) \right] \\
&\geq \delta^t \cdot \left[D_w(w_n; w_n) \cdot \frac{N_{w_n}(w_n; w_n)}{D_{w_n}(w_n; w_n)} - N_w(w_n; w_{n+1}) \right] \\
&> \delta^t \cdot \left[D_w(w_n; w_n) \cdot \frac{N_w(w_n; w_n)}{D_w(w_n; w_n)} - N_w(w_n; w_{n+1}) \right] \\
&\geq \delta^t \cdot \left[D_w(w_n; w_n) \cdot \frac{N_w(w_n; w_n)}{D_w(w_n; w_n)} - N_w(w_n; w_n) \right] \\
&= 0
\end{aligned}$$

Now suppose that $w_n \geq \max\{w, w_{n-1}\}$. Then we have $D_w(w_n; w_{n-1}) \leq D_w(w_n; w_n) \leq D_{w_n}(w_n; w_n) < 0$. Hence, we have

$$\begin{aligned}
Q_w(w_n; w_{n-1}, w_{n+1}) &= \delta^t \cdot \left[D_w(w_n; w_{n-1}) \cdot \frac{N_{w_n}(w_n; w_n)}{D_{w_n}(w_n; w_n)} - N_w(w_n; w_{n+1}) \right] \\
&\leq \delta^t \cdot \left[D_w(w_n; w_t) \cdot \frac{N_{w_n}(w_n; w_n)}{D_{w_n}(w_n; w_n)} - N_w(w_n; w_{n+1}) \right] \\
&\leq \delta^t \cdot \left[D_w(w_n; w_t) \cdot \frac{N_w(w_n; w_n)}{D_w(w_n; w_n)} - N_w(w_n; w_{n+1}) \right] \\
&< \delta^t \cdot \left[D_w(w_n; w_t) \cdot \frac{N_w(w_n; w_n)}{D_w(w_n; w_n)} - N_w(w_n; w_n) \right] \\
&= 0.
\end{aligned}$$

This completes the proof of (ii).

(iii) Finally, assume $w = w_{n+1}$. Suppose that $w_n \leq \min\{w_{n+1}, w_{n-1}\}$. If $D_w(w_n; w_{n-1}) \geq 0$, we will have $Q_w(w_n; w_{n-1}, w_{n+1}) > 0$. If $D_w(w_n; w_{n-1}) < 0$, we will have $D_{w_n}(w_n; w_n) \leq D_w(w_n; w_n) \leq D_w(w_n; w_{n-1}) < 0$, and hence,

$$Q_w(w_n; w_{n-1}, w_{n+1}) = \delta^t \cdot \left[D_w(w_n; w_{n-1}) \cdot \frac{N_{w_n}(w_n; w_n)}{D_{w_n}(w_n; w_n)} - N_w(w_n; w_{n+1}) \right]$$

$$
\begin{aligned}
&\geq \ \delta^t \cdot \left[D_w(w_n; w_n) \cdot \frac{N_{w_n}(w_n; w_n)}{D_{w_n}(w_n; w_n)} - N_w(w_n; w_{n+1}) \right] \\
&\geq \ \delta^t \cdot \left[D_w(w_n; w_n) \cdot \frac{N_w(w_n; w_n)}{D_w(w_n; w_n)} - N_w(w_n; w_{n+1}) \right] \\
&\geq \ \delta^t \cdot \left[D_w(w_n; w_n) \cdot \frac{N_w(w_n; w_n)}{D_w(w_n; w_n)} - N_w(w_n; w_n) \right] \\
&= \ 0
\end{aligned}
$$

Note that the second and third inequalities are strict if $w_n < w_{n+1} = w$. Hence, we have $Q_w(w_n; w_{n-1}, w_{n+1}) < 0$ if $w_n < w$ and $w_n \leq \min\{w_{n+1}, w_{n-1}\}$.

Now suppose that $w_n \geq \max\{w, w_{n-1}\}$. Then we have $D_w(w_n; w_{n-1}) \leq D_w(w_n; w_n) \leq D_{w_n}(w_n; w_n) < 0$. Hence, we have

$$
\begin{aligned}
Q_w(w_n; w_{n-1}, w_{n+1}) &= \ \delta^t \cdot \left[D_w(w_n; w_{n-1}) \cdot \frac{N_{w_n}(w_n; w_n)}{D_{w_n}(w_n; w_n)} - N_w(w_n; w_{n+1}) \right] \\
&\leq \ \delta^t \cdot \left[D_w(w_n; w_t) \cdot \frac{N_{w_n}(w_n; w_n)}{D_{w_n}(w_n; w_n)} - N_w(w_n; w_{n+1}) \right] \\
&\leq \ \delta^t \cdot \left[D_w(w_n; w_t) \cdot \frac{N_w(w_n; w_n)}{D_w(w_n; w_n)} - N_w(w_n; w_{n+1}) \right] \\
&\leq \ \delta^t \cdot \left[D_w(w_n; w_t) \cdot \frac{N_w(w_n; w_n)}{D_w(w_n; w_n)} - N_w(w_n; w_n) \right] \\
&= \ 0
\end{aligned}
$$

Note that the second and third inequalities are strict if $w_n > w_{n+1} = w$. Hence, we have $Q_w(w_n; w_{n-1}, w_{n+1}) < 0$ if $w_n > w$ and $w_n \geq \max\{w, w_{n-1}\}$. This completes the proof of (iii). ■

Using these lemmas, we now show that optimal N-deviations are either monotone or constant.

Lemma 6 *Let $w^* = \{w_n^*\}_{n=1}^N$ be any solution to the optimization problem*

$$
\max_{\tilde{w}} S_N(\tilde{w}; w_{N+1}) \tag{6}
$$

(i) If $w < w_{N+1}$ then $w < w_1^* < \ldots < w_N^* < w_{N+1}$.

(ii) If $w > w_{N+1}$ then $w > w_1^* > \ldots > w_N^* > w_{N+1}$.

(iii) If $w = w_{N+1}$ then $w = w_1^* = \ldots = w_N^* = w_{N+1}$.

Proof The proof is by induction on N. When $N = 1$, we have $w_{N-1} = w_0 = w$, and aim to find the optimal w_1 given w_2.

(i) When $w < w_2$, part (i) of Lemma 5 tells us that $Q_w(w_1; w_0, w_2) > 0$ when $w_1 \leq w$ and that $Q_w(w_1; w_0, w_2) < 0$ when $w_1 \geq w_2$. Hence, the optimal w_1 lies in (w, w_2).

(ii) When $w > w_2$, part (ii) of Lemma 5 tells us that $Q_w(w_1; w_0, w_2) > 0$ when $w_1 \leq w_2$ and that $Q_w(w_1; w_0, w_2) < 0$ when $w_1 \geq w$. Hence, the optimal w_1 lies in (w_2, w).

(iii) When $w = w_2$, part (iii) of Lemma 5 tells us that $Q_w(w_1; w_0, w_2) > 0$ when $w_1 < w$ and that $Q_w(w_1; w_0, w_2) < 0$ when $w_1 > w_2$. Hence, the optimal w_1 must be w.

This proves the lemma when $N = 1$.

Now suppose Lemma 6 holds at $N-1$; i.e., namely any sequence $\{w_n^*\}_{n=1}^{N-1}$ that maximizes $S_{N-1}(w_1, \ldots, w_{N-1}; w_N)$ is strictly monotone or constant. We want to prove that any sequence $\{w_n^*\}_{n=1}^{N}$ that maximizes $S_N(w_1, w_2, \ldots, w_N; w_{N+1})$ is also strictly monotone or constant. We consider the three cases in turn.

(i) Consider the case with $w < w_{N+1}$.

- Assume that the optimal $w_N^* \leq w$. Then the optimal sequence $\{w_n^*\}_{n=1}^{N-1}$ that maximizes $S_{N-1}(w_1, \ldots, w_{N-1}; w_N)$ must satisfy $w \geq$

$w_1^* \geq \cdots \geq w_{N-1}^* \geq w_N^*$. So we have $w_N^* \leq w_{N-1}^* \leq w < w_{N+1}$. Part (i) of Lemma 5 tells us that $Q_w(w_N^*; w_{N-1}^*, w_{N+1}) > 0$. This is contradictory to the fact that w_N^* is optimal.

- Assume that the optimal $w_N^* \geq w_{N+1}$. Then the optimal sequence $\{w_n^*\}_{n=1}^{N-1}$ that maximizes $S_{N-1}(w_1, \ldots, w_{N-1}; w_N)$ must satisfy $w < w_1^* < \cdots < w_{N-1}^* < w_N^*$. So we have $w_N^* \geq w_{N+1}$ and $w_N^* > w_{N-1}^*$. Part (i) of Lemma 5 tells us that $Q_w(w_N^*; w_{N-1}^*, w_{N+1}) < 0$. This is contradictory to the fact that w_N^* is optimal.

In sum: when $w < w_{N+1}$, the optimal w_N^* must lie in (w, w_{N+1}). Therefore, the sequence $\{w_n^*\}_{n=1}^{N}$ that maximizes $S_N(w_1, w_2, \ldots, w_N; w_{N+1})$ must satisfy $w < w_1^* < \cdots < w_{N-1}^* < w_N^* < w_{N+1}$.

(ii) Consider the case with $w > w_{N+1}$.

- Assume that the optimal $w_N^* \geq w$. Then the optimal sequence $\{w_n^*\}_{n=1}^{N-1}$ that maximizes $S_{N-1}(w_1, \ldots, w_{N-1}; w_N)$ must satisfy $w \leq w_1^* \leq \cdots \leq w_{N-1}^* \leq w_N^*$. So we have $w_N^* \geq w_{N-1}^* \geq w > w_{N+1}$. Part (ii) of Lemma 5 tells us that $Q_w(w_N^*; w_{N-1}^*, w_{N+1}) < 0$. This is contradictory to the fact that w_N^* is optimal.

- Assume that the optimal $w_N^* \leq w_{N+1}$. Then the optimal sequence $\{w_n^*\}_{n=1}^{N-1}$ that maximizes $S_{N-1}(w_1, \ldots, w_{N-1}; w_N)$ must satisfy $w > w_1^* > \cdots > w_{N-1}^* > w_N^*$. So we have $w_N^* \leq w_{N+1}$ and $w_N^* < w_{N-1}^*$. Part (ii) of Lemma 5 tells us that $Q_w(w_N^*; w_{N-1}^*, w_{N+1}) > 0$. This is contradictory to the fact that w_N^* is optimal.

In sum: when $w > w_{N+1}$, the optimal w_N^* must lie in (w_{N+1}, w). Therefore, the sequence $\{w_n^*\}_{n=1}^N$ that maximizes $S_N(w_1, w_2, \ldots, w_N; w_{N+1})$ must satisfy $w > w_1^* > \cdots > w_{N-1}^* > w_N^* > w_{N+1}$.

(iii) Finally, consider the case with $w = w_{N+1}$.

- Assume that the optimal $w_N^* > w$. Then the optimal sequence $\{w_n^*\}_{n=1}^{N-1}$ that maximizes $S_{N-1}(w_1, \ldots, w_{N-1}; w_N)$ must satisfy $w < w_1^* < \cdots < w_{N-1}^* < w_N^*$. So we have $w_N^* > w_{N-1}^* > w = w_{N+1}$. Part (iii) of Lemma 5 tells us that $Q_w(w_N^*; w_{N-1}^*, w_{N+1}) < 0$. This is contradictory to the fact that w_N^* is optimal.
- Assume that the optimal $w_N^* < w$. Then the optimal sequence $\{w_n^*\}_{n=1}^{N-1}$ that maximizes $S_{N-1}(w_1, \ldots, w_{N-1}; w_N)$ must satisfy $w > w_1^* > \cdots > w_{N-1}^* > w_N^*$. So we have $w_N^* < w_{N-1}^* < w = w_{N+1}$. Part (iii) of Lemma 5 tells us that $Q_w(w_N^*; w_{N-1}^*, w_{N+1}) > 0$. This is contradictory to the fact that w_N^* is optimal.

In summary, when $w = w_{N+1}$, the optimal w_N^* must be w. Therefore, the sequence $\{w_n^*\}_{n=1}^N$ that maximizes $S_N(w_1, w_2, \ldots, w_N; w_{N+1})$ must satisfy $w = w_1^* = \cdots = w_{N-1}^* = w_N^* = w_{N+1}$.

This concludes the proof of the lemma. ■

On the basis of these lemmas we can now complete the proof of Theorem 1. If G is not a SSAE then some worker w has a profitable deviation. Lemma 2 guarantees that worker w has a profitable finite deviation in which worker w produces the sequence of outputs $G(w_1), G(w_2), \ldots, G(w_N)$ and then produces $G(w)$ forever after. However Lemma 6 tells us that among all N-deviations that end with $w_{N+1} = w$, the optimal one is constant. That is, there is no profitable deviation, and G is a steady state assortative equilibrium, as asserted. This completes the proof of Theorem 1.

Proof of Theorem 2 Note that for values of the payment coefficient λ in any closed interval $[a, b] \subset (0, 1)$, the right-hand side of the ODE (1) is uniformly Lipschitz in λ. It follows from the general theory of first-order ODE's that the (unique) strictly increasing solutions to (1) are continuous in λ on every such closed interval, and hence on the whole open interval $(0, 1)$. In particular, firm profit (in the unique SSAE) is continuous in λ on the whole open interval $(0, 1)$. Moreover, firm profit is strictly positive for every $\lambda \in (0, 1)$. As $\lambda \to 0$, the payments received by workers tends to 0 so the effort exerted by workers and the output produced also tend to 0, so firm profit tends to 0. As $\lambda \to 1$, the effort exerted by workers and the output produced remain bounded, so firm profit again tends to 0. It follows that firm profit is continuous on the whole closed interval $[0, 1]$. Hence the firm has an optimal profit-maximizing choice λ^*, and an equilibrium exists. This completes the proof of Theorem 2.

Proof of Theorem 3 Because firm profit is simply a fraction of total output, in order to show that $\Pi_{\text{random}} < \Pi_{\text{assort}}$ it is sufficient to show that when workers choose the myopically optimal output $Y^*(w, t)$ the matching that maximizes total output (and hence firm profit) is assortative. To show this, it is sufficient (and in fact necessary) to verify that $Y^*(w, t)$ is supermodular; i.e., that the mixed partial $\frac{\partial^2 Y^*}{\partial w \partial t} > 0$. To do this we first need to compute partials of e^*.

The myopically optimal effort $e^*(w, t)$ is defined by the first order condition

$$\lambda p(w)q(t) - k(w)c'\left[e^*(w,t)\right] = 0. \tag{7}$$

Implicit differentiation shows that

$$\begin{aligned}
\frac{\partial e^*(w,t)}{\partial w} &= \frac{\lambda p'(w)q(t) - k'(w)c'\left[e^*(w,t)\right]}{k(w)c''\left[e^*(w,t)\right]} \\
\frac{\partial e^*(w,t)}{\partial t} &= \frac{\lambda p(w)q'(t)}{k(w)c''\left[e^*(w,t)\right]} \\
\frac{\partial^2 e^*(w,t)}{\partial w \partial t} &= \frac{\lambda q'(t)\left[p'(w)k(w) - p(w)k'(w)\right]\left[1 - \frac{c'[e^*(w,t)]c'''[e^*(w,t)]}{\{c''[e^*(w,t)]\}^2}\right]}{\left[k(w)\right]^2 c''\left[e^*(w,t)\right]}
\end{aligned}$$

By definition, $Y^*(w,t) = Y\left[e^*(w,t), w, t\right]$, so we differentiate to obtain:

$$\begin{aligned}
\frac{\partial^2 Y^*(w,t)}{\partial w \partial t} &= p(w)q(t)\frac{\partial^2 e^*(w,t)}{\partial w \partial t} + p'(w)q'(t)e^*(w,t) \\
&+ p(w)q'(t)\frac{\partial e^*(w,t)}{\partial w} + p'(w)q(t)\frac{\partial e^*(w,t)}{\partial t}.
\end{aligned}$$

Our computations of the derivatives of e^* and our assumptions (especially log-concavity of c') guarantee that each of the terms on the right-hand side is positive so we conclude that Y^* is supermodular. As we have noted this guarantees that $\Pi_{\text{random}} < \Pi_{\text{assort}}$.

To see that $\Pi_{\text{assort}} < \Pi_{\text{SSAE}}$ consider the ODE (1). Note that the denominator has the form $q(w)F(w)$ and that $F(w) = 0$ is precisely the first-order condition for myopically optimal choice of effort for worker w when matched with task w. At the SSAE, $G'(w)$ and the numerator of the ODE are strictly positive so $F(w)$ is also strictly positive; it follows that at the SSAE, worker w is exerting *more* than the myopically optimal effort and hence producing output *greater* than $Y^*(w,w)$. Because profit is a fixed fraction of output, the firm obtains greater output from each worker at the SSAE than when matching is assortative and workers choose myopically optimal effort. In particular, $\Pi_{\text{assort}} < \Pi_{\text{SSAE}}$.

Finally, to see that $\Pi_{\text{SSAE}} < \Pi_{\text{FI}}$ note that in the full information setting the firm induces the effort level that leaves the worker with 0 net utility. In the SSAE, each worker obtains strictly positive utility. (Any worker who obtained 0 utility could simply produce slightly less output in the current period and exert 0 effort in the future, thereby obtaining strictly positive utility.). Hence, in the full information setting, workers exert greater effort than in SSAE and hence produce greater output and greater profit for the firm, so $\Pi_{\text{SSAE}} < \Pi_{\text{FI}}$. This completes the proof of Theorem 3.

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